STRESSES IN BEAMS

MET 4501 — LECTURE NOTES — PROF. LEAH GINSBERG

DEFINITIONS

YOUNG'S MODULUS (MODULUS OF ELASTICITY)

A MEASURE OF THE STIFFNESS OF A MATERIAL. IN THE LINEAR ELASTIC REGION,

$$\sigma = E\varepsilon$$

(HOOKE'S LAW)*

σ : STRESS
E : YOUNG'S MODULUS
ε : ENGINEERING STRAIN

POISSON'S RATIO

A MEASURE OF THE PROPORTIONAL DECREASE IN DIMENSION WHEN A MATL IS STRETCHED IN ANOTHER DIMENSION

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}}$$

$\varepsilon_{lateral}$ ← PERPENDICULAR TO F
ε_{axial} ← IN DIRECTION OF F

STRESS

THE INTERNAL FORCE PER UNIT AREA WITHIN A MATERIAL.

$$\sigma = \frac{F}{A}$$

σ : STRESS
F : APPLIED FORCE
A : CROSS-SECTIONAL AREA OVER WHICH F IS APPLIED

STRAIN

THE MEASURE OF DEFORMATION REPRESENTING THE DISPLACEMENT BETWEEN PARTICLES IN A MATERIAL.

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0}$$

ε : ENGINEERING STRAIN
l : DEFORMED LENGTH
l_0 : ORIGINAL LENGTH

*HOOKE'S LAW FOR SHEAR IS

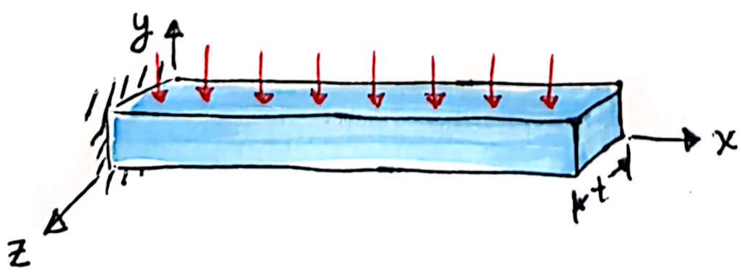
$$\tau = G\gamma$$

τ : SHEAR STRESS
G : SHEAR MODULUS
γ : SHEAR STRAIN

RECALL THAT PLANE STRESS OCCURS WHEN STRESSES ON ONE SURFACE ARE ZERO. FOR EXAMPLE, σ_z, τ_{xz}, τ_{zx}, τ_{yz}, AND $\tau_{zy} = 0$.

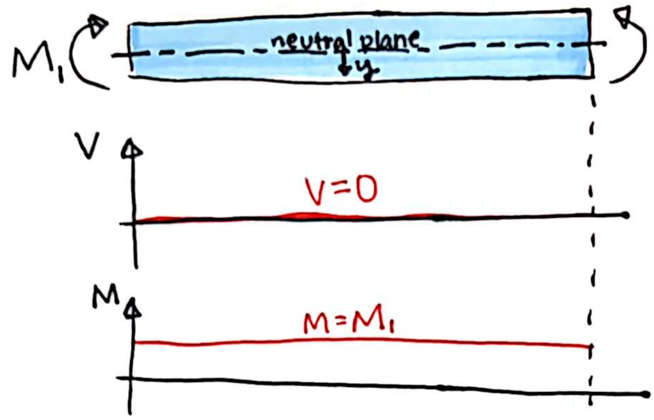
SIMILARLY, PLANE STRAIN OCCURS WHEN STRAINS IN ONE DIRECTION ARE ZERO. FOR EXAMPLE, ε_z, γ_{yz}, γ_{zy}, γ_{xz}, $\gamma_{zx} = 0$

TAKE THE BEAM DRAWN BELOW, FOR EXAMPLE. AS THE THICKNESS (t) APPROACHES 0, THE BEAM APPROACHES A STATE OF PLANE STRESS. AS THE THICKNESS (t) APPROACHES ∞, THE BEAM APPROACHES A STATE OF PLANE STRAIN.



NORMAL STRESSES FOR BEAMS IN BENDING

LET'S CONSIDER A BEAM IN PURE BENDING. ELEMENTS OF MATERIAL ON TOP OF THE BEAM WILL BE IN COMPRESSION, WHILE ELEMENTS OF MATERIAL ON THE BOTTOM WILL BE IN TENSION.



IN THE MIDDLE, THERE IS A PLANE WHICH CONTAINS MATL ELEMENTS WITH ZERO BENDING STRESS. THIS IS CALLED THE $\boxed{\text{NEUTRAL PLANE.}}$

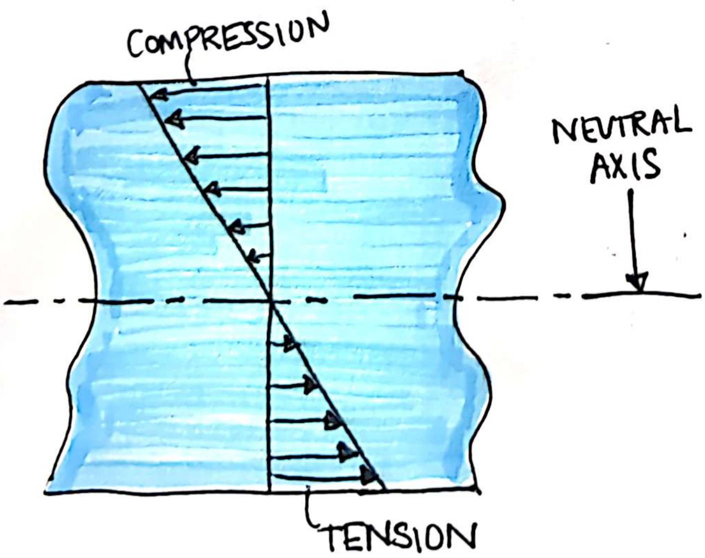
BENDING STRESS VARIES LINEARLY WITH DISTANCE FROM THE NEUTRAL PLANE (OR AXIS), AND IS GIVEN BY THE FLEXURE FORMULA:

$$\boxed{\sigma = \frac{My}{I}}$$

WHERE I IS THE SECOND AREA MOMENT ABOUT THE Z AXIS.

$$I = \int y^2 dA$$

LOOKING AT A SECTION OF THE BEAM IN PURE BENDING,



THE MAXIMUM MAGNITUDE OF BENDING OCCURS WHERE y IS GREATEST. DENOTE THIS MAX. MAGNITUDE OF y AS c, TO WRITE

$$\sigma_{max} = \frac{Mc}{I}$$

WHICH IS OFTEN WRITTEN AS

$$\sigma_{max} = \frac{M}{Z}$$

$Z = I/c$ IS THE SECTION MODULUS

A NOTE ON TWO-PLANE BENDING: IF BENDING OCCURS IN TWO PLANES, SAY xy AND xz, THE INDUCED BENDING STRESSES ARE ADDITIVE:

$$\sigma_x = \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

SHEAR STRESSES FOR BEAMS IN BENDING

MOST BEAMS HAVE BOTH SHEAR FORCES AND BENDING MOMENTS.

THE AVERAGE SHEAR STRESS (τ) ACTING OVER A CROSS-SECTIONAL AREA (A) IS

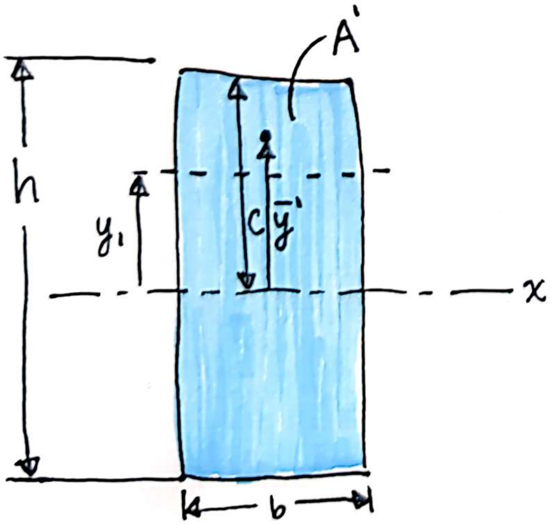
$$\tau = \frac{V}{A}$$

THE SHEAR STRESS ON A FREE SURFACE (i.e. THE TOP & BOTTOM OF THE BEAM) MUST BE ZERO. THE DISTRIBUTION OF SHEAR STRESSES ACTING ON A CROSS-SECTION IS GIVEN BY

$$\boxed{\tau = \frac{VQ}{Ib}}$$

(SEE SECTION 3-11 IN SHIGLEY FOR THE DERIVATION, WHICH USES EQUILIBRIUM)

WHERE Q IS THE FIRST MOMENT OF AREA FOR THE PORTION OF THE CROSS-SECTION ABOVE THE LOCATION THAT WE WANT TO CALCULATE τ FOR.



$$Q = \int_{y_1}^{c} y\,dA = \bar{y}' A'$$

WHERE $\bar{y}'$ IS THE DISTANCE FROM THE NEUTRAL PLANE TO THE CENTROID OF THE AREA A'.

FOR EXAMPLE, FOR A RECTANGULAR CROSS-SECTION

$$Q = \frac{b}{2}(c^2 - y_1^2)$$

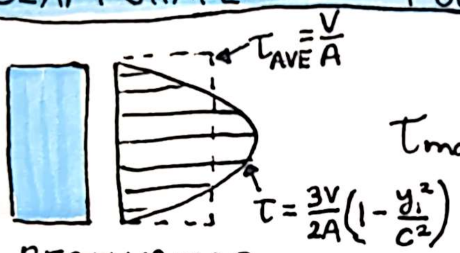
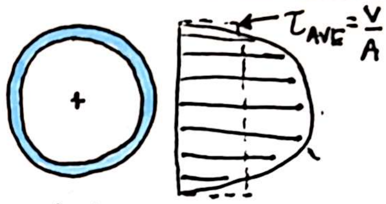
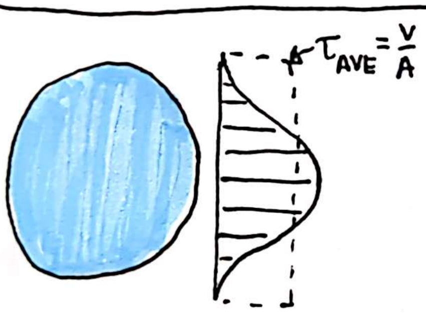
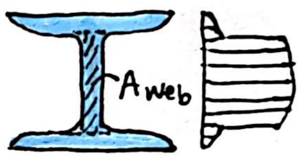
SUBSTITUTING IN $I = bh^3/12$ (FOR A RECTANGULAR CROSS-SECTION) AND THE EXPRESSION FOR Q INTO THE EQUATION FOR SHEAR STRESS:

$$\tau = \frac{VQ}{Ib} \Rightarrow \tau = \frac{3V}{2A}\left(1 - \frac{y_1^2}{c^2}\right)$$

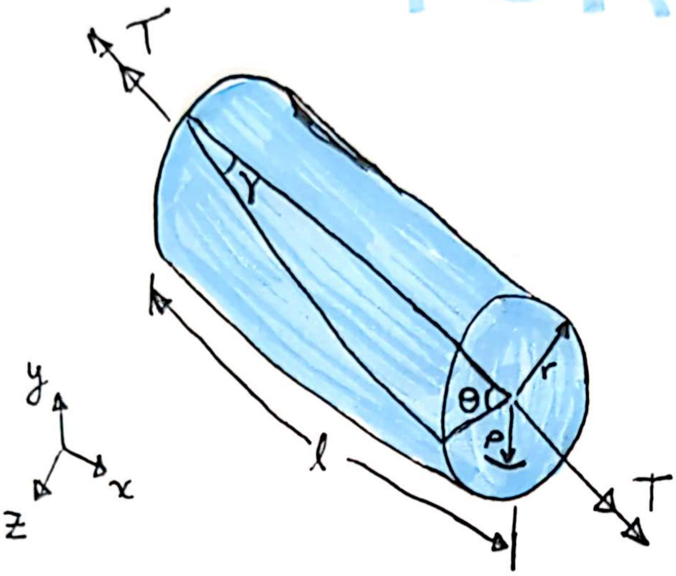
MAXIMUM SHEAR STRESS OCCURS WHEN $y_1 = 0$ (AT THE BENDING NEUTRAL AXIS)

$$\tau_{max} = \frac{3V}{2A} \quad \text{(FOR A RECTANGULAR CROSS-SECTION)}$$

FORMULAS FOR MAXIMUM TRANSVERSE SHEAR STRAIN:

BEAM SHAPE	FORMULA	BEAM SHAPE	FORMULA
 RECTANGULAR	$\tau_{max} = \frac{3V}{2A}$	 HOLLOW, THIN-WALLED ROUND	$\tau_{max} = \frac{2V}{A}$
 CIRCULAR	$\tau_{max} = \frac{4V}{3A}$	 STRUCTURAL I BEAM (THIN-WALLED)	$\tau_{max} \approx \frac{V}{A_{web}}$

TORSION



FOR A SOLID ROUND BAR SUBJECT TO A TORQUE (T) THAT IS COLLINEAR WITH ITS AXIS IS SAID TO BE IN TORSION.

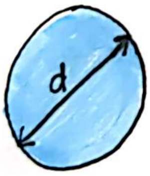
THE ANGLE OF TWIST FOR A SOLID ROUND BAR IS

$$\boxed{\theta = \frac{T\ell}{GJ}}$$

WHERE G IS THE SHEAR MODULUS OF THE MATERIAL, AND J IS THE POLAR MOMENT OF INERTIA. (OR THE ~~SECO~~ POLAR SECOND MOMENT OF AREA)

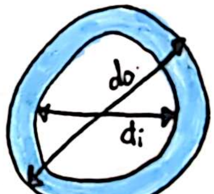
THE POLAR MOMENT OF INERTIA (J) DEFINES THE RESISTANCE OF A CROSS-SECTION TO TORSIONAL DEFORMATION, DUE ONLY TO THE SHAPE OF THE CROSS-SECTION. ($J = \int_0^r \rho^2 dA$)

FOR A SOLID ROUND SECTION,



$$J = \frac{\pi d^4}{32}$$

FOR A HOLLOW ROUND SECTION,



$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

FOR SMALL SHEAR STRAINS (γ) WE CAN SOLVE FOR THE SHEAR STRAINS USING SIMPLE TRIGONOMETRY.

$$\gamma \approx \tan\gamma = \frac{r\theta}{\ell}$$

(ON THE OUTER SURFACE OF THE BAR)

AT ANY RADIAL DISTANCE FROM THE CENTER OF THE BAR (ρ),

$$\boxed{\gamma = \frac{\rho\theta}{\ell}}$$

THE SHEAR STRESS DUE TO TORSION IS GIVEN BY

$$\boxed{\tau = \frac{T\rho}{J}}$$